

Testing General Relativity With Laser Accelerated Electron Beams

L. Á. Gergely^{*,†} and T. Harko^{**}

^{*}*Department of Theoretical Physics, University of Szeged, Szeged 6720, Tisza L. krt. 84, Hungary*

[†]*Department of Experimental Physics, University of Szeged, 6720 Szeged, Dóm tér 9, Hungary*

^{**}*Department of Physics and Center for Theoretical and Computational Physics, The University of Hong Kong, Pok Fu Lam Road, Hong Kong, P. R. China*

Abstract. Electron accelerations of the order of $10^{21} g$ obtained by laser fields open up the possibility of experimentally testing one of the cornerstones of general relativity, the weak equivalence principle, which states that the local effects of a gravitational field are indistinguishable from those sensed by a properly accelerated observer in flat space-time. We illustrate how this can be done by solving the Einstein equations in vacuum and integrating the geodesic equations of motion for a uniformly accelerated particle.

Keywords: high intensity lasers, electrons, equivalence principle, space-time singularities

PACS: 42.60.Jf, 41.75.Ht, 04.20.-q, 04.20.Dw

INTRODUCTION

Acceleration of electrons by intense laser pulses is of fundamental interest in view of potential scientific and technological applications like modeling astrophysical phenomena, realizing table-top particle accelerators, and fast ignition in inertial confinement fusion [1]. A properly injected fast electron with charge e and mass m_e is captured by the laser field and accelerated to high energies when the laser has an intensity parameter $a_0 = eE_0/m_e\omega c \geq 100$, with E_0 and ω the electrical field and frequency, respectively [2]. At the Lawrence Berkeley National Laboratory electrons were accelerated with an experimental laser plasma accelerator to 1 GeV, corresponding to an acceleration of $5.4 \times 10^{20} g$, over about 3.3 cm [3], while with a plasma wakefield accelerator electrons achieved $8.9 \times 10^{20} g$, gaining 42 GeV energy over 85 cm [4]. Here $g = 9.81 \text{ m/s}^2$ is the gravitational acceleration on the surface of the Earth. For comparison, the gravitational acceleration on the surface of a neutron star with mass $M = 3M_\odot$ and radius $R = 10 \text{ km}$ is only $a \approx GM/R^2 = 4.08 \times 10^{11} g$. Such a stellar object is quite close to the black hole limit of general relativity.

The prospect of producing electron accelerations of the order of $10^{21} g$ by using laser fields opens up the possibility of direct experimental test of the weak equivalence principle, which states that "The local effects of motion in a curved space-time are indistinguishable from those of an accelerated observer in a flat space" [5]. During the acceleration process, the space-time around the electron may become curved, thus an equivalent gravitational field may be generated. As a first approximation we consider the motion of the accelerated particle in a general relativistic framework. In Section 2 we solve the *vacuum* Einstein gravitational field equations, obtaining the space-time

metric. The electrons move along the geodesics of this vacuum space-time. In Section 3 these equations of motion are integrated, and the solution is given in parametric form. We briefly discuss our results and point out possible generalizations in Section 4.

THE METRIC IN THE PRESENCE OF THE UNIFORMLY ACCELERATED PARTICLE

The inertial motion of an object is contained in the variational principle $\delta \int dS_i = 0$, with $dS_i^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$ the square of the Minkowski line element, taken in the reference frame $x^\alpha = (ct, x, y, z)$. The uniformly accelerated motion is described by a similar variational principle, $\delta \int dS_a = 0$. As the constant acceleration three-vector singles out a preferred direction, say the z -axis, the line element squared can be chosen as

$$dS_a^2 = F(z)c^2 dt^2 - \frac{1}{F(z)} dz^2 - dx^2 - dy^2, \quad (1)$$

with $F(z)$ a function of the z -coordinate only, to be determined by solving Einstein's gravitational field equations.

By introducing the 4-velocity $\dot{x}^\alpha = dx^\alpha/dS_a$, the variational principle can be rewritten conveniently as

$$\delta \int \sqrt{g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta} dS_a = 0. \quad (2)$$

The Euler-Lagrange equations corresponding to this variational problem are

$$\frac{d}{dS_a} \left(\frac{\partial}{\partial \dot{x}^\mu} \sqrt{g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta} \right) - \frac{\partial}{\partial x^\mu} \sqrt{g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta} = 0. \quad (3)$$

After some simple transformations Eq. (3) gives the geodesic equation,

$$\frac{d^2 x^\mu}{dS_a^2} + \Gamma_{\alpha\beta}^\mu \dot{x}^\alpha \dot{x}^\beta = 0, \quad (4)$$

where $\Gamma_{\alpha\beta}^\mu = \frac{1}{2} g^{\mu\sigma} (g_{\alpha\sigma,\beta} + g_{\beta\sigma,\alpha} - g_{\alpha\beta,\sigma})$ are the Christoffel symbols associated to the metric, and the comma denotes the partial derivative with respect to the respective coordinate. For the metric given by Eq. (1), $g_{\mu\nu,\lambda} = F' \delta_{0\mu} \delta_{0\nu} \delta_{3\lambda} + (F'/F^2) \delta_{3\mu} \delta_{3\nu} \delta_{3\lambda}$ and the Christoffel symbols become

$$\Gamma_{\alpha\beta}^\mu = \frac{1}{2} \left[\frac{F'}{F} (\delta^{0\mu} \delta_{3\alpha} \delta_{3\beta} + \delta^{0\mu} \delta_{0\alpha} \delta_{0\beta} - \delta^{3\mu} \delta_{3\alpha} \delta_{3\beta}) + F F' \delta^{3\mu} \delta_{0\alpha} \delta_{0\beta} \right]. \quad (5)$$

Next we explore Einstein's field equations in vacuum $R_{\mu\nu} = 0$, where $R_{\mu\nu}$ is the Ricci curvature tensor¹

$$R_{\mu\nu} = \Gamma_{\mu\rho,\nu}^\rho - \Gamma_{\mu\nu,\rho}^\rho + \Gamma_{\mu\rho}^\sigma \Gamma_{\sigma\nu}^\rho - \Gamma_{\mu\nu}^\sigma \Gamma_{\sigma\rho}^\rho. \quad (6)$$

¹ We follow the metric and curvature conventions of Ref. [6].

With the explicit form of the Christoffel symbols taken into account the Ricci tensor can be written as

$$R_{\mu\nu} = (\ln \sqrt{-g})_{,\mu\nu} - \Gamma_{\mu\nu}^{\sigma} (\ln \sqrt{-g})_{,\sigma} - \Gamma_{\mu\nu,\rho}^{\rho} + \Gamma_{\mu\rho}^{\sigma} \Gamma_{\sigma\nu}^{\rho} . \quad (7)$$

Since for the considered metric $\sqrt{-g} = 1$, the differential equations giving $F(x)$ take the simple form

$$\frac{\partial \Gamma_{\mu\nu}^{\rho}}{\partial x^{\rho}} = \Gamma_{\mu\rho}^{\sigma} \Gamma_{\sigma\nu}^{\rho} . \quad (8)$$

Exploring Eq. (5) we obtain

$$\left(\frac{F'}{F}\right)^2 = -\left(\frac{F'}{F}\right)' , \quad F'^2 = (FF')' , \quad (9)$$

which are solved for

$$F(x) = C_0 + C_1 z, \quad (10)$$

with C_0 and C_1 arbitrary constants of integration. By requiring that at $z = 0$ the metric is manifestly Minkowskian, the metric compatible with a uniformly accelerated particle becomes

$$dS_a^2 = \left(1 - \frac{2az}{c^2}\right) c^2 dt^2 - \frac{dz^2}{1 - 2az/c^2} - dx^2 - dy^2 , \quad (11)$$

with $a = d^2 z/dt^2$ the constant, non-relativistic acceleration of the particle (this can be seen from the classical limit of the Lagrangian). Despite a manifest singularity at $z = c^2/2a$, the metric is flat.

THE EQUATION OF MOTION OF UNIFORMLY ACCELERATED PARTICLES

From Eq. (2) one can read the Lagrangian, or equivalently, its square, leading to identical equations of motion:

$$\Lambda = \left(1 - \frac{2az}{c^2}\right) c^2 \dot{t}^2 - \frac{1}{1 - 2az/c^2} \dot{z}^2 \equiv 1 . \quad (12)$$

Here $\dot{t} = dt/dS_a$ and $\dot{z} = dz/dS_a$, respectively. We obtain

$$\frac{d}{dS_a} \left(\frac{\partial \Lambda}{\partial \dot{z}} \right) = -\frac{2}{1 - 2az/c^2} \ddot{z} - \frac{2a}{c^2} \frac{1}{(1 - 2az/c^2)^2} \dot{z}^2 , \quad (13)$$

$$\frac{\partial \Lambda}{\partial z} = -\frac{2a}{c^2} \left[\frac{1}{1 - 2az/c^2} + \frac{2}{(1 - 2az/c^2)^2} \dot{z}^2 \right] . \quad (14)$$

In the second equation we have eliminated \dot{t}^2 in favor of \dot{z}^2 , by employing the second equality in (12). The Euler-Lagrange equation simplifies to

$$\frac{d^2 z}{dS_a^2} = \frac{a}{c^2} , \quad (15)$$

showing that z is quadratic in S_a . By choosing the initial conditions $z(0) = 0$ and $(dz/dS_a)|_{(t=0, z=0)} = 0$, we obtain the parametric solution

$$z(t) = \frac{c^2}{2a} \left(\frac{e^{2at/c} - 1}{e^{2at/c} + 1} \right)^2, \quad S_a(t) = \frac{c^2}{a} \frac{e^{2at/c} - 1}{e^{2at/c} + 1}. \quad (16)$$

At large accelerations $e^{2at/c} \gg 1$, $z \rightarrow c^2/2a$ and $dz/dS_a = (e^{2at/c} - 1) / (e^{2at/c} + 1) \rightarrow 1$. This is when the metric becomes singular. For a particle moving at a constant acceleration of $a = 2 \times 10^{20}$ cm/s² (achievable by laser fields [3, 4]), this happens quite fast, at $z = 4.5$ cm.

DISCUSSIONS AND FINAL REMARKS

Laser physics experiments generating large accelerations could in principle produce intense equivalent gravitational fields, even black holes. In this paper we have explored the metric and free motion of a test particle in vacuum. The uniformly accelerated particle then leads to a flat space-time, with a manifest singularity at $z = c^2/2a$. A more realistic description of the acceleration of electron by laser beams could be achieved by including the accelerating electromagnetic field with energy-momentum tensor $T_{\mu\nu}$ as a source of the Einstein equation, $R_{\mu\nu} = (8\pi G/c^2) T_{\mu\nu}$, in this case the space-time will not be flat; also modifying correspondingly the equation of motion, $m_0 c(du^\mu/ds + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta) = (e/c) u_\nu F^{\mu\nu}$. This case deserves future investigation.

ACKNOWLEDGMENTS

LÁG was supported by the Project "TÁMOP-4.2.1/B-09/1/KONV-2010-0005 - Creating the Center of Excellence at the University of Szeged" of the European Union, and co-financed by the European Regional Development Fund, also by the Hungarian Scientific Research Fund (OTKA) grant no. 81364.

REFERENCES

1. Z.-M. Sheng, L.-W. Zhu, M. Y. Yu, and Z.-M. Zhang, New Journal of Physics **12**, 013011 (2010).
2. A. Bahari and V. D. Taranukhin, Laser Phys. Lett. **1**, 397 (2004).
3. W. P. Leemans, B. Nagler, A. J. Gonsalves, Cs. Toth, K. Nakamura, C. G. R. Geddes, and E. Esarey, Nature Physics **418**, 696 (2006).
4. I. Blumenfeld, C. E. Clayton, F.-J. Decker, M. J. Hogan, C. K. Huang, R. Ischebeck, R. Iverson, C. Joshi, T. Katsouleas, N. Kirby, W. Lu, K. A. Marsh, W. B. Mori, P. Muggli, E. Oz, R. H. Siemann, D. Walz, and M. Zhou, Nature **445**, 741 (2007).
5. R. H. Dicke, Annals of Physics **31**, 235 (1965).
6. L. D. Landau and E. M. Lifshitz, The Classical Theory of Fields, Butterworth Heinemann, Oxford (1998).